MATH 732: CUBIC HYPERSURFACES

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1. The number 27

See the disclaimer section.

Let us start by assembling a number of facts we have proved, hinted at, given as exercises, or asserted.

(1) Let X be a smooth projective variety with a rank r vector bundle $\mathcal{E}$. If $s \in \mathrm{H}^0(\mathcal{E})$ is a section that meets the zero section transversely, then $c_r(\mathcal{E}) = [(s = 0)] \subseteq \mathrm{H}^{2r}(X, \mathbf{Z})$.

(2) The Fano scheme of a degree d hypersurface $X = (F = 0) \subseteq \mathbf{P}^{n+1}$ is the scheme theoretic zero locus

$$(s_F = 0) \subseteq \mathrm{Gr}(2, n+2)$$

where $s_F \in \mathrm{H}^0(\mathrm{Gr}(2, n+2), \mathrm{Sym}^d(\mathcal{S}^\vee))$ is the induced section.

(3) If $X = (F = 0) \subseteq \mathbf{P}^{n+1}$ is a smooth cubic hypersurface, then it's Fano scheme:

$$\mathrm{F}(X) \subseteq \mathrm{Gr}(2, n+2)$$

is smooth of the expected dimension $2n-4$. (In particular, the induced section s_F of $\mathrm{Sym}^3(\mathcal{S}^\vee)$ meets the zero section transversely).

Proposition 1.1. *The Grassmannian* $\mathbf{G} = \mathrm{Gr}(2,4) \subseteq \mathbf{P}^5$ *is a quadric hypersurface defined by the Plücker equation:*

$$X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0.$$

For the (dual of the) tautological bundle $\mathcal{S}^\vee$ *we have:*

$$c_1(\mathcal{S}^\vee) = [H_{\mathbf{G}}] \quad \text{and} \quad c_2(\mathcal{S}^\vee) = [\Lambda] \subseteq \mathbf{G}$$

where $\Lambda \subseteq \mathbf{G}$ *is a 2-dimensional linear space.*

Proof. Recall that the Grassmannian $\mathbf{G}$ can be parametrized by 2×4 matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

such that the not all the minors $X_{ij} = \det A_{ij}$ vanish. These minors give the Plücker embedding.

The dual of the tautological sequence gives the quotient

$$V^\vee \otimes_k \mathcal{O}_{\mathbf{G}} \to \mathcal{S}^\vee \to 0.$$

Taking $\wedge^2$ gives the Plücker embedding as well. This shows that

$$c_1(\det(\mathcal{S}^\vee)) = c_1(\det(\mathcal{S}^\vee)) = [H_{\mathbf{G}}] \in \mathrm{H}^2(\mathbf{G}, \mathbf{Z}).$$

Finally, $\mathcal{S}^\vee$ is globally generated, so the zero locus of a general section computes $c_2(\mathcal{S}^\vee)$. The sections are determined by linear forms

$$\lambda : V \to k.$$

The section λ vanishes at a 2-plane $W \subseteq V$ if and only if $\lambda|_W \equiv 0$. In other words, the zero locus

$$(\lambda = 0) \subseteq \mathbf{G}$$

paramatetrizes planes in V contained in $(\lambda = 0) \simeq k^3 \subseteq V$. This is a linearly embedded $\mathbf{P}^2$. □

Lemma 1.2. *If $\mathcal{E}$ is a rank 2 vector bundle on X, then*

$$c_4(\operatorname{Sym}^3 \mathcal{E}) = 9c_2(\mathcal{E})(2c_1(\mathcal{E})^2 + c_2(\mathcal{E})).$$

Proof. This is of course an application of the *splitting principle*: which says that there's no harm in assuming $\mathcal{E}$ is split when computing the Chern classes $\operatorname{Sym}^3 \mathcal{E}$.

Let's write: $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$. Then

$$c_1(\mathcal{E}) = a + b \quad \text{and} \quad c_2(\mathcal{E}) = ab.$$

Such a splitting would give rise to a splitting

$$\operatorname{Sym}^3(\mathcal{E}) = \mathcal{O}(3a) \oplus \mathcal{O}(2a+b) \oplus \mathcal{O}(a+2b) \oplus \mathcal{O}(3b).$$

So we see

$$\begin{aligned} c_4(\operatorname{Sym}^3(\mathcal{E})) &= 3a(2a+b)(a+2b)3b \\ &= 9ab(2(a+b)^2 + ab) \\ &= 9c_2(\mathcal{E})(2c_1(\mathcal{E})^2 + c_2(\mathcal{E})). \end{aligned}$$

□

Theorem 1.3. *Every smooth cubic surface contains exactly 27 lines.*

Proof. We know by our previous comments that for a smooth cubic surface $X = (F = 0) \subseteq \mathbf{P}^3$, the Fano scheme of X:

$$F(X) = (s_F = 0) \subseteq \mathrm{Gr}(2,4)$$

is smooth and 0-dimensional, so a finite collection of points (i.e. lines). We also know:

$$[F(X)] = c_4(\operatorname{Sym}^3(\mathcal{S}^\vee)) \in \mathrm{H}^4(\mathrm{Gr}(2,4), \mathbf{Z}).$$

So the *degree* of $c_4(\mathrm{Sym}^3(\mathcal{S}^\vee))$ equals this number of lines.

By Lemma 1.2, we just need to compute

(1) $c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))c_1(\mathrm{Sym}^3(\mathcal{S}^\vee))^2$, and
(2) $c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))^2$.

The first is just the degree of the linearly embedded $\mathbf{P}^2$ that represents $c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))$, i.e. 1. To compute $c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))^2$ amounts to asking how many planes are contained in the intersection of 2 hyperplanes in V. The answer is again 1. (In both of these answers we are using the projection formula for these Chern classes.)

So finally we have:

$$\begin{aligned}\begin{pmatrix}\text{number of}\\ \text{lines in } X\end{pmatrix} &= \deg([F(X)]),\\ &= 9(2c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))c_1(\mathrm{Sym}^3(\mathcal{S}^\vee)) + c_2(\mathrm{Sym}^3(\mathcal{S}^\vee))),\\ &= 9(2\cdot 1+1) = 27.\end{aligned}$$

□

Remark 1.4. There are many proofs of this theorem to varying degrees of generality. In another direction, Segre [Seg42] computed the possible number of real lines in a smooth cubic surface:

$$X \subseteq \mathbf{P}^3_{\mathbf{R}}.$$

Note, that as X is a real surface, the set of lines is defined over $\mathbf{R}$, so any complex lines must come in conjugate pairs. As the total number of complex lines is 27, this guarantees the existence of at least 1 real line.

Segre broke the real lines into two types (based on a natural *relative orientation* $\mathrm{Sym}^3\mathcal{S}^\vee$) *elliptic lines* and *hyperbolic lines*:

(ell. lines)	(hyper. lines)	(total lines)
12	15	27
6	9	15
2	5	7
0	3	3

This shows that smooth real cubic surfaces always contain at least 3 lines! (Note, the difference between the number of types of lines is always 3.) There are other results over non-closed fields as well, see e.g. [KW21].

Remark 1.5. If $X = (x^3+y^3+z^3+w^3=0) \subseteq \mathbf{P}^3$ is the Fermat cubic surface, it is possible to just write down all 27 lines. Let $\zeta, \omega \in k$ be any two cubed root of -1. Then

$$L_{\zeta,\omega} = (y-\zeta x = w - \omega z = 0) \subseteq \mathbf{P}^3.$$

is contained in X. This gives 9 lines, and the rest can be found from the permutation action of $\mathfrak{S}_4$ on X.

Exercise 1. *Prove that the if $\mathcal{E}$ is a globally generated vector bundle on a projective variety X then a general section $s \in \mathrm{H}^0(X, \mathcal{E})$ meets the zero section transversely.*

Exercise 2. *Count the number of lines in a general quintic threefold $X \subseteq \mathbf{P}^4$.*

Exercise 3. *Count the number of lines in a general septic (degree 7) fourfold $X \subseteq \mathbf{P}^5$.*

References

[KW21] Jesse Leo Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Compos. Math.*, 157(4):677–709, 2021.

[Seg42] B. Segre. *The Non-singular Cubic Surfaces.* Oxford University Press, Oxford, 1942.